5. V. M. Zhdanov, "Baro- and thermodiffusion in a capillary," Zh. Prikl. Mekh. Tekh. Fiz., No. 2, 48-51 (1982).
6. V. M. Zhdanov and V. A. Zaznoba, "Nonisothermal flow of a rarefied gas in a circular channel," Inzh.-Fiz. Zh., 44, No. 5, 772-779 (1983).
7. M. N. Kogan, Rarefied Gas Dynamics [in Russian], Nauka, Moscow (1967).
8. S. Chapman and T. Cowling, Mathematical Theory of Non-Uniform Gases, Oxford Univ. Press (1970).
9. V. Zhdanov, Yu. Kagan, and A. Sezykin, "Effect of external momentum transfer on diffusion in a gas mixture," Zh. Eksp. Teor. Fiz., 42, No. 3, 858-867 (1962).
10. H. Grad, "On the kinetic theory of rarefied gases," Commun. Pure Appl. Math., 2, No. 4, 331-407 (1949).
11. S. K. Loyalka, "Approximate method in the kinetic theory," Phys. Fluids, 14, No. 11, 2291-2294 (1971).
12. S. R. de Groot and P. Mazur, Non-Equilibrium Thermodynamics, Elsevier (1962).

WAVE PROPERTIES AND SHEAR STRESS OF A TURBULENT BOUNDARY LAYER

Ya. A. Vagramenko

UDC 532.525.2

The wave theory of turbulence [1-3] is applied to the problem of a turbulent boundary layer near a planar wall. Preliminary results earlier published have been refined.

In a turbulent flow the statistical ensemble state of large-scale vortices is described by the equation [2]

$$ih\frac{\partial\psi}{\partial t} = -\frac{h^2}{2\rho}\nabla^2\psi,\ i=\sqrt{-1}, \tag{1}$$

where ∇^2 is the Laplacian. The special representation $\psi = a \exp ib$ makes it possible to obtain from (1) an expression for the energy of motion $h\omega$ and an equation for the probability flow a^2 for stationary turbulence ($\partial a^2/\partial t = 0$):

$$\omega = \frac{1}{2}\rho U^2 - \frac{h^2}{2\rho}\frac{\nabla^2 a}{a}, \tag{2}$$

$$h\,\mathrm{div}\,(a^2\,\mathrm{grad}\,b) = 0. \tag{3}$$

In this case $h|\mathrm{grad}\ b| = \rho U$, $\omega = \partial b/\partial t$. The negative term on the right-hand side of Eq. (2) reflects the statistical aspect of vortex-particle interactions, and equals the fluctuation energy $a^2\rho U^2/2$. Thus,

$$\nabla^2 a + |\mathrm{grad}\,b|^2 a^3 = 0 \tag{4}$$

and, besides, $h\omega = (1 + a^2)\rho U^2/2$. Since $h = \rho U|\mathrm{grad}\ b|^{-1}$, it follows that

$$\omega = \frac{1+a^2}{2}U\,|\mathrm{grad}\,b|. \tag{5}$$

The amplitude a coincides with the local turbulence intensity u'/U, where u' is the fluctuation in translational velocity. The representation of kinetic properties of vortex-particles in terms of wave characteristics implies that the individual motion of vortices is expressed in terms of statistical ensemble properties, thus forming a set of vortex-particles. The probability distribution of the amplitude a is such that in the region of wave existence

$$\int a^2 db = 1. \tag{6}$$

Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 45, No. 3, pp. 410-419, September, 1983. Original article submitted April 5, 1982.

In the boundary layer approximation, when $\partial/\partial y \gg \partial/\partial x$, one obtains from (3), (4)

$$aa'' + c^2 = 0, \tag{7}$$

$$a^2 b' = c, \tag{8}$$

where c is an integration constant. The differentiation in (7), (8) is with respect to the variable $\varphi = y/\delta$.

In the boundary region of a boundary layer the fluctuations are subject to action of viscous stresses from the side of the sublayer. At a free boundary layer turbulence is similar to a jet. Therefore the vortex structures in these two regions are nonidentical, as is the case for the behavior of probability waves. A vortex is attached in the boundary region of the wave (with its plane almost parallel to the wall), and becomes "free" in the exterior region, where its plane is perpendicular to the wall [5]. For the coexistence of "antagonistic" vortices of two kinds it is necessary to have a buffer wave zone, which is neutral in the sense that there exists no predominant vortex direction in it.

On the whole the fluctuation intensity must decrease away from the wall. In the buffer zone, however, the disorder in the vortex orientation leads to a general balance in fluctuation intensities (probabilities), and therefore the wave amplitude will be constant over the width of this zone (a similar circumstance is pointed out in [3]). The limiting high level reached at $da/d\varphi = 0$, will be the fluctuation level immediately at the wall, while for the exterior part of the wave – it is the level at the buffer zone.

Under the conditions mentioned the configuration across the standing wave corresponds to a solution of Eqs. (7), (8) of the form

$$\varphi = \frac{a_0}{c}\sqrt{\frac{\pi}{2}}\,\Phi(r),\ r^2 = \ln\frac{a_0}{a},\ \varphi \leqslant \varphi_1, \tag{9}$$

$$\Phi(r) = \frac{2}{\sqrt{\pi}}\int_0^r \exp(-r^2)\,dr,$$

$$\varphi = \varphi_2 + \frac{a_1}{c}\sqrt{\frac{\pi}{2}}\,\Phi(r),\ r^2 = \ln\frac{a_1}{a},\ \varphi \geqslant \varphi_2, \tag{10}$$

$$a = a_1,\ \varphi_1 \leqslant \varphi \leqslant \varphi_2. \tag{11}$$

The coordinates φ_1, φ_2 refer, respectively, to the exterior boundary of the inhomogeneity region of the wave and to the internal boundary of the exterior region of the wave, also inhomogeneous. The values of the constant c in (9), (10) are, in principle, different. The buffer region occupies the interval $\varphi_1 \leq \varphi \leq \varphi_2$, and in it $a = a_1$. Immediately near the wall $a = a_0$. Since the level a_1 is characteristic of the boundary region,

$$a_1 = a_0 \exp(-r_1^2), \tag{12}$$

where $r_1^2 = \ln(a_0/a_1)$ occurs at $\varphi = \varphi_1$. The interval $0 \leq \varphi \leq \varphi_1$ corresponds to the existence region of the associated vortices, and is therefore uniquely determined as half the wavelength of the probability standing wave, for which the points $\varphi = 0$ and $\varphi = \varphi_1$ are nodes. In the interval $\varphi_1 \leq \varphi \leq \varphi_2$, in which the wave motion of vortices is stable at the limiting level of probability fluctuations, the parameter h, related to vortex circulation [2], also reaches the limiting value $h = h_1$. Under these conditions, not included within the special form of the solution of (3), (4) for inhomogeneous waves, the standing wave is determined on the basis of (1), with $\psi = \psi_0 \exp(-i\omega t)$, where ψ_0 is a function of coordinates. Since $\partial\psi/\partial t = -i\omega\psi$, then, according to (1),

$$\frac{h_1}{2\rho}\nabla^2\psi + \omega h_1 \psi = 0. \tag{13}$$

Using relation (5) in (13), within the boundary layer approximation we have

$$\psi'' + (1 + a_1^2) b'^2 \psi = 0 \quad (\varphi_1 \leqslant \varphi \leqslant \varphi_2). \tag{14}$$

It has been taken into account that for a boundary layer $\rho U \sim h\partial b/\partial y$, $\nabla^2\psi \sim \partial^2\psi/\partial y^2$. If in the buffer region $h_1 = \mathrm{const}$, we obtain for it $\partial b/\partial y = \rho U/h_1$, while, according to Eq. (8), $h_1 = c^{-1}\rho\delta U_1 a_1^2$, corresponding to the h value at the boundary $\varphi = \varphi_1$. Consequently, in this region $b' = cU/U_1 a_1^2$, where U_1 is the velocity at the boundary $\varphi = \varphi_1$. It is hence seen that to avoid a discontinuity in the phase gradient at $\varphi = \varphi_2$ the c values in (10) and (9) must be chosen in such a manner that their ratio equal U_2/U_1, where U_2 is the velocity at the boundary $\varphi = \varphi_2$.

Assuming that the velocity near the wall increases quickly almost until the exterior flow velocity U_δ, for $\varphi > \varphi_1$ we take $U/U_1 \approx 1$; then $b' = c/a_1^2$ in the region $\varphi_1 \leq \varphi \leq \varphi_2$, and, consequently, the total phase change is here

$$b_1 = \frac{c}{a_1^2}(\varphi_2 - \varphi_1). \tag{15}$$

In this approximation the constant c is the same through the whole layer width. We now find from Eq. (14) $\psi = \sin[c(1+a_1^2)^{1/2} a_1^{-2}(\varphi - \varphi_1)]$. By the existence condition of a standing wave we must have $\psi = 0$ at the nodes $\varphi = \varphi_1$, $\varphi = \varphi_2$; therefore,

$$\frac{c}{a_1^2}(1+a_1^2)^{\frac{1}{2}}(\varphi_2 - \varphi_1) = n\pi, \quad n = 1, 2, 3, \ldots \tag{16}$$

Considering (15) and (16) simultaneously, we obtain

$$b_1 = (1+a_1^2)^{-1/2} n\pi. \tag{17}$$

Thus, tuning of the wave motion in the exterior zone $\varphi > \varphi_2$ to the vortex fluctuation regime in the boundary zone $\varphi \leq \varphi_1$ requires matching of intensities and amount of half wavelengths in the buffer zone.

It can be seen that

$$\frac{db}{dr} = \frac{\sqrt{2}}{a_+}\exp r^2, \tag{18}$$

where $a_+ = a_0$ for $\varphi \leq \varphi_1$, and $a_+ = a_1$ for $\varphi \geq \varphi_2$. We also take into account (11) and the condition at the exterior boundary $\varphi = 1$, $a = 0$, $r = \infty$. By integrating then, for $a = a_+\exp(-r^2)$, over the layer width we obtain from (6)

$$\sqrt{\frac{\pi}{2}}\,[a_1 + a_0\Phi(r_1)] + a_1^2 b_1 = 1. \tag{19}$$

In a stationary boundary the half wavelength phase variation is π, and then, according to (18), $a_0 = D(r_1)\sqrt{2}/\pi$, where $D(r) = \int_0^r \exp r^2 dr$. This a_0 value must be used in (12), which gives $a_1 = \sqrt{2}\pi^{-1}D(r_1)\exp(-r_1^2)$. With account of (17) we now obtain (19) in the form

$$\frac{2n}{\pi}D^2(r_1)\left[1 + \frac{2}{\pi^2}D^2(r_1)\exp(-2r_1^2)\right]^{1/2}\exp(-2r_1^2) + \frac{1}{\sqrt{\pi}}D(r_1)[\Phi(r_1) + \exp(-r_1^2)] = 1. \tag{20}$$

By fixing n the quantity r_1 is found as the root of Eq. (20), following which we determine a_0, a_1. Integrating Eq. (8) over the layer width, we see that the normalization condition (6) must also correspond to the relation $c(\varphi_1 - \varphi_2 + 1) + a_1^2 b_1 = 1$. With account of Eq. (15) we find from the latter $c = 1$.

Under the conditions $r = \infty$, $\varphi = 1$ we determine from (10) the separation boundary of the buffer and exterior wave zones $\varphi_2 = 1 - a_1 c^{-1}\sqrt{\pi/2}$. From (10) we also find the exterior nodes of the first standing half wavelength for $\varphi > \varphi_2$, immediately adjacent to the buffer zone: $\varphi_3 = \varphi_2 + a_1 c^{-1}\sqrt{\pi/2}\Phi(r_3)$, where the r_3 value is related to the wave intensity at the point $\varphi = \varphi_3$ and is found from (18) under the condition that the phase change be π on the interval $\varphi_3 - \varphi_2$. In that case $\sqrt{2}D(r_3)/a_1 = \pi$, and after eliminating a_1 we obtain the condition $D(r_3) = D(r_1)\exp(-r_1^2)$. We note that for $\varphi = \varphi_3$, $a = a_3 = a_1\exp(-r_3^2)$. The r_1, r_3 values for varying n are shown in Table 1. The characteristic wave structures are shown in Table 2, from which it is seen that the turbulence intensity a_0 and the width of the wave boundary region decrease with increasing n.

Our problem, relating the wave motion with the regular flow, is described by the equations

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \rho^{-1}\frac{\partial \tau}{\partial y}, \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \tag{21}$$

The shear stress τ, generated by the action of turbulent fluctuations, which for an inhomogeneous structure of probability waves are such that exchange by fluctuation weights of kinetic energy between different flow layers is not mutually compensating. As a result the single fluctuation on a closed contour becomes an exciting vortex element of length l, within whose limits is generated a fluctuation rate $l\,\partial u/\partial y$, while the circulation change

TABLE 1. Values of the Constants r_1, r_3

n	1	2	3	4	5	6	7	8
r_1	0,88	0,73	0,64	0,56	0,51	0,47	0,44	0,41
r_3	0,5	0,48	0,46	0,43	0,41	0,38	0,37	0,35

TABLE 2. Characteristic Wave Structures

n	φ_1	φ_2	φ_3	a_0	a_1	a_3
1	0,515	0,7	0,85	0,525	0,243	0,19
2	0,35	0,71	0,85	0,400	0,232	0,184
3	0,26	0,72	0,85	0,333	0,221	0,179
4	0,2	0,74	0,85	0,280	0,204	0,17
5	0,17	0,76	0,86	0,251	0,193	0,163
6	0,14	0,77	0,86	0,225	0,181	0,157
7	0,12	0,78	0,86	0,211	0,174	0,151
8	0,10	0,80	0,87	0,193	0,163	0,143

equals $l^2\partial u/\partial y$. The statistical equivalent of the latter quantity is the mathematical expectation, defined on the set of all segments l inclined to the vortex formations, consisting of the contour length. The pulsation rate occurring on the contour is commensurate with the "shear rate" $(\tau/\rho)^{1/2}$. The inclination to the fluctuation generated at the contour is characterized by the probability a^2. For a contour length on the order of the large vortex scale L we obtain a mathematical expectation of a "circulation quantum" of order $a^2 L(\tau/\rho)^{1/2}$. Consequently,

$$a^2 L\sqrt{\tau/\rho} = \xi_1 l^2 \partial u/\partial y, \tag{22}$$

in agreement with the coefficient ξ_1 found from the compatibility requirement of the regular and wave motion fields.

Starting from a condition of the type of the uncertainty relation, vortex scales were found [2], according to which

$$L = \frac{1}{2}\left|\frac{\partial b}{\partial x}\right|^{-1}(1+a^2)^{-1/2},\ l = \frac{1}{2}\left|\frac{\partial b}{\partial y}\right|^{-1}(1+a^2)^{-1/2}.$$

For slow longitudinal flow changes an "almost self-similar" solution is possible of Eq. (21) in the current function $\delta U_\delta f(\varphi)$, such that $u = U_\delta f'$, $v = \gamma U_\delta(\varphi f' - f)$, $\gamma = d\delta/dx$, where γ is a weakly varying function of x. Keeping in mind that $|\partial b/\partial x| = \gamma\varphi b'/\delta$, $|\partial b/\partial y| = b'/\delta$, where $b' = c/a^2$, we obtain from (22) $\tau = \xi\gamma^2 c^{-2}\rho U_\delta^2(\varphi f'')^2$. Here the coefficient $\xi = {}^1/_4\xi_1^2(1+a^2)^{-1}$ is taken to be a constant quantity, since $a^2 \ll 1$. Eliminating it, and using conditions at the viscous sublayer boundary $y = y_0$, we have on it $u = u_0$, $\partial u/\partial y = u_0\delta/\varphi_0$, with $\varphi_0 = y_0/\delta$. Denoting $m = u_0/U_\delta$, at the point $\varphi = \varphi_0$ we have $f' = m$, $f'' = m/\varphi_0$. Introducing then the definition $\tau_0 = c_f\rho U_\delta^2/2$, for $\varphi = \varphi_0$, $\tau = \tau_0$, we find $\xi = c_f c^2/2m^2\gamma^2$, and consequently, $\tau = c_f\rho U_\delta^2(\varphi f'')^2/2m^2c^2$. Taking into account the latter expression, from (21) we obtain the equation

$$\varphi^2 f''' + \varphi f'' + \beta f = 0, \tag{23}$$

in which

$$\beta = \gamma^2 m^2 c^2/c_f. \tag{24}$$

The solution of Eq. (23) is represented in the form of a power series in φ, also containing terms with the logarithmic factor:

$$f = s_0 + s_1\varphi + s_2\varphi^2 + \ldots + (c_0 + c_1\varphi + c_2\varphi^2 + \ldots)\ln\frac{\varphi}{\varphi_0}. \tag{25}$$

For smallest divergence we take in the latter $f = 0$ for $\varphi = \varphi_0$. As a result of substituting (25) into (23) we obtain $s_0 = 0$, $c_0 = 0$, and relations for determining the other coefficients: $5c_2 + 2s_2 + \beta s_1 = 0$, $s_1 = -\varphi_0 s_2$, $2c_2 = -\beta c_1$. For $\beta\varphi_0 \ll 2$ the solution (25) acquires the form

$$f = c_1\left[\frac{5}{4}\beta\varphi(\varphi-\varphi_0) + \varphi\left(1 - \frac{\beta}{2}\varphi\right)\ln\frac{\varphi}{\varphi_0}\right],$$

whence, neglecting the small quantity $\frac{5}{4}\beta\varphi_0$, we obtain

$$f' = c_1\left[1 + 2\beta\varphi + (1 - \beta\varphi)\ln\frac{\varphi}{\varphi_0}\right]. \tag{26}$$

Since $f' = m$ for $\varphi = \varphi_0 \to 0$, one must put $c_1 = m$.

The parameter β is determined from the conditions at the exterior layer boundary. As in the case of jet turbulence [2, 3], we start from the assumption that the exterior boundary of the regular flow in the layer coincides with the site $\varphi = \varphi_3$ of the exterior half-wave, within whose limits there exists a large scale exterior vortex adjacent to the buffer zone, and the following damped half-waves ($\varphi_3 \le \varphi \le 1$) have no direct relation to the shear flow. This representation corresponds to the condition $\varphi = \varphi_3$, $f' = 1$. At the layer edge the velocity transforms continuously to the velocity of the unperturbed flow, i.e., $\varphi = \varphi_3$, $f'' = 0$. The last of these conditions gives $\beta\varphi_3 = [\ln(\varphi_3/\varphi_0) - 1]^{-1}$, in which case we obtain from the first condition

$$\alpha^2 - \left(1 + \frac{1}{m}\right)\alpha + 1 + \frac{1}{m} = 0, \quad \alpha \equiv \ln\frac{\varphi_3}{\varphi_0}. \tag{27}$$

The solution of Eq. (27), satisfying the obvious requirement of unconfined sublayer thinning for $m \to 0$, is written in the form

$$\alpha = (m + m_1)/2m, \quad m_1 = 1 + \sqrt{1 - 2m - 3m^2}. \tag{28}$$

Consequently, $\beta\varphi_3 = 2m/(m_1 - m)$. By comparing the latter expression with (24) we find the change in the boundary layer width

$$\gamma = \frac{d\delta}{dx} = \frac{2c_f}{m\varphi_3(m_1 - m)}. \tag{29}$$

The velocity profile (26) is

$$\frac{u}{U_\delta} = f' = m\left[1 + \frac{4m}{m_1 - m}\frac{\varphi}{\varphi_3} + \left(1 - \frac{2m}{m_1 - m}\frac{\varphi}{\varphi_3}\right)\left(\ln\frac{\varphi}{\varphi_3} + \frac{m + m_1}{2m}\right)\right]. \tag{30}$$

Having determined the boundary conditions at a smooth wall, we start from the idea of discontinuous structure of a viscous sublayer. The latter is developed as nonstationary viscous flow in the period between two fluctuations of the boundary vortex. This viscous flow breaks down completely as the vortex approaches the wall. If the nonstationary sublayer is formed by sudden flow generation with some velocity u_c, practically constant within the longitudinal vortex scale, the velocity distribution in this viscous flow obeys the dependence [6]

$$\frac{u}{u_c} = \Phi(\varepsilon), \quad \varepsilon = \frac{y}{2}\sqrt{\frac{\rho}{t\mu}}. \tag{31}$$

For a relatively narrow sublayer ($y_0/\delta \ll 1$) we use $u/u_c \simeq \varepsilon$ (the velocity profile in the sublayer is linear). Taking into account expression (5), for $|\mathrm{grad}\, b| \simeq \partial b/\partial y$, $u \simeq U$ we determine the period $T = 2\pi/\omega$ of the sublayer restoration in the fluctuation frequency near the wall:

$$T = \frac{4\pi}{u_c(1 + a_0^2)}\left(\frac{\partial b}{\partial y}\right)^{-1}. \tag{32}$$

For $y \to 0$, u_c must be expressed here from the mean velocity distribution (30), since the "viscous" and "turbulent" velocity profiles near the wall are completely linked. Since for $\varphi \to 0$ we have $\partial b/\partial y = c/a_0^2\delta$, for $t = T$ we find from (32) and (31) the sublayer width

$$y_0 = 4\varepsilon_0\left[\frac{\pi a_0^2\delta\mu}{c\rho u_c(1 + a_0^2)}\right]^{1/2}. \tag{33}$$

In (33) ε_0 corresponds to some velocity u_0 at the sublayer boundary at the moment of its vortex breakdown. By comparing the expressions of the flow stress at the wall $\tau_0 = \mu u_0/y_0$ and $\tau_0 = \frac{1}{2}c_f\rho U_\delta^2$, it follows that

$$\mathrm{Re} = 2m/\varphi_0 c_f, \tag{34}$$

where $Re = \delta \rho U_\delta / \mu$ is the Reynolds number, determined by the width of the fluctuation region. Taking also into account that $\varepsilon_0 \simeq u_0/u_c$, we obtain from (33)

$$\varphi_0 = \frac{8\pi a_0^2 c_f}{c(1 + a_0^2) m^2 m_c^2}. \tag{35}$$

Here $m_c = u_c/u_0$. The velocity u_c of the flow, forming a nonstationary viscous flow, is realized by the statistics at a distance y_c from the wall equal to the interval of transverse vortex localization for $y \to 0$ (see above), i.e., $y_c = \frac{1}{2}(\partial b/\partial y)^{-1}(1 + a_0^2)^{-1/2}$. This corresponds to $\varphi_c = y_c/\delta = \frac{1}{2}a_0^2(1 + a_0^2)^{-1/2}$. Then, according to (30), $m_c = N/m$, where

$$N = \frac{m_1 + 3m}{2} + \frac{A(3m^2 - mm_1)}{m_1 - m} + \left(m - \frac{2m^2 A}{m_1 - m}\right) \ln A;$$

$$A = \frac{a_0^2}{2\varphi_3 (1 + a_0^2)^{1/2}}.$$

According to (27) $\varphi_0 = \varphi_3 \exp(-\alpha)$. Comparing this expression with (35), we obtain (for c = 1)

$$c_f = \frac{\varphi_3 (1 + a_0^2)}{8\pi a_0^2} \frac{N^3}{m} \exp(-\alpha). \tag{36}$$

From (34) and (35) we find

$$Re = \frac{1 + a_0^2}{4\pi a_0^2 c_f^2} N^3. \tag{37}$$

The functions (36), (37) determine the friction law in the boundary layer parametrically in terms of the velocity ratio $m = u_c/U_\delta$ (Fig. 1a,b). One may note the presence of a narrow region of Re numbers, in which one observes an initial increase of the friction coefficient during layer thickening (Fig. 2a). In this region the velocity ratio m_c at the center of the oscillating boundary vortex to that at the sublayer boundary increases from approximately 1.3 to 2.0. In this case one observes an increase in c_f and γ, which reflects the transition regime of formation of a turbulent boundary layer.

For sudden (exponential) perturbation increase at the final transition stage [6] the width of the turbulent boundary layer (within the mean flow) will be practically the same as the width δ_1 of the laminar layer up to the start of transition, i.e., $\delta_1 = \varphi_3 \delta$. In the laminar boundary layer $Re_1 = 5(Re_x)^{1/2}$ [6], where Re_1 and Re_x are the Reynolds numbers determined by the width δ_1 and by the longitudinal coordinate x. Consequently, the start of the transition corresponds to $Re_{x0} = (\frac{1}{5}\varphi_3 Re_0)^2$, where Re_{x0} and Re_0 correspond to Re_x and Re at $m_c \to 1$, i.e., the start of an established turbulent structure. Further, along the boundary layer the relation between the numbers Re and Re_x is determined from the dependence

$$Re_x = Re_{x0} + \int_{Re_0}^{Re} \gamma^{-1} d\,Re,$$

in which the slope γ of the fluctuation boundary region is known from (29) (Fig. 2b). Table 3 shows the boundary layer parameters at the start of the transition (a) and after the completion of transition (b), which corresponds to a maximum of c_f, γ. The change in the local friction coefficient as a function of Re_x is shown in Fig. 3.

Each time, following the achievement of a new critical layer width there occurs a rearrangement of the fluctuation field beneath in the flow for n increasing by unity. There exists a number of "standard" structures of a turbulent boundary layer, so that its start for n = 2, 3, 4, ... corresponds to one of the numbers $Re_{x0} = 2.25 \cdot 10^5$, $6.63 \cdot 10^5$, $1.71 \cdot 10^6$, and so forth (Table 3). The well-known experimental data [6] indicate that the observed Re_{x0} of the transition are indeed grouped near these calculated values. (For n = 1 a realization of a stable structure is, obviously, impossible, since the width of the buffer zone is smaller than the size of the boundary vortex penetrating this zone.) Which of the transitions mentioned above is realized depends on the level of the initial perturbation. This problem requires separate consideration [4].

The external fluctuation boundary layer is such that the probability wave "splashes" following the statistical limits of the mean flow $\varphi = \varphi_3$. According to Table 2, $\varphi_3 = 0.85$–0.86, i.e., the region of possible vortex neglect is approximately 1.2 times wider than the region of regular motion in the layer. This corresponds to experimental data.

TABLE 3. Data for Establishment of Turbulent Structure

n		m	10^{-4} Re	10^{-5} Re$_x$	$10^3 c_f$	$10^2 \varphi_3 \gamma$
1	а	0,325	0,127	0,59	6,11	5,16
	б	0,310	0,156	0,65	6,30	4,81
2	а	0,31	0,28	2,25	3,53	2,70
	б	0,265	0,39	2,62	4,5	3,21
3	а	0,28	0,48	6,63	3,04	2,18
	б	0,24	0,686	7,44	3,72	2,71
4	а	0,255	0,768	17,1	2,65	1,90
	б	0,225	1,04	18,3	3,16	2,35
5	а	0,245	1,01	30,4	2,3	1,65
	б	0,21	1,49	32,7	2,87	2,18
6	а	0,23	1,35	54,1	2,20	1,6
	б	0,20	2,01	57,6	2,62	2,04
7	а	0,225	1,58	73,8	2,05	1,51
	б	0,195	2,37	78,1	2,49	1,97
8	а	0,215	1,98	117	1,96	1,47
	б	0,190	2.86	122	2,33	1,87

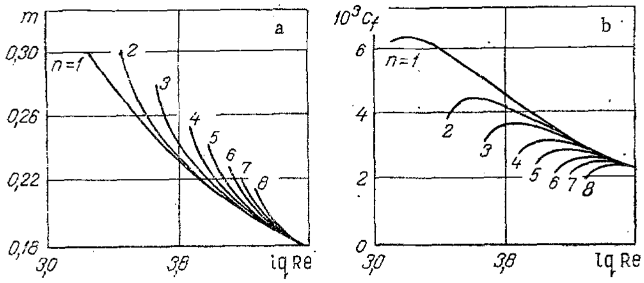

Fig. 1. The local friction coefficient (a) and the velocity at the sublayer boundary (b) as a function of Re.

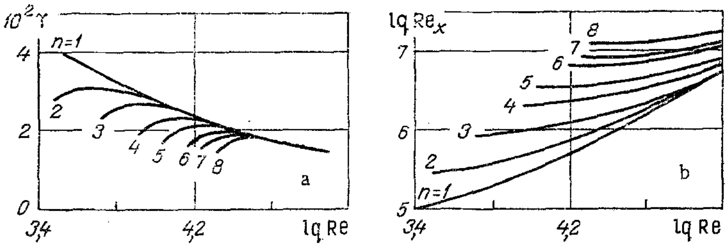

Fig. 2. The slope of the external boundary layer (a) and the corresponding Reynolds number (b).

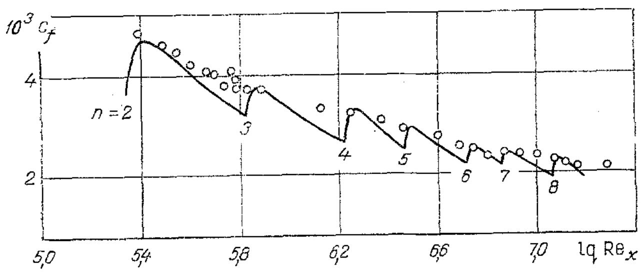

Fig. 3. Change in the local friction coefficient along the boundary layer.

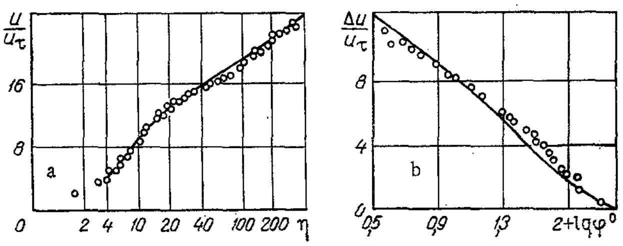

Fig. 4. Velocity variation near a wall (a) and across the width of the boundary layer (b).

Figure 4a shows the profile of u/u_τ according to (30) for m = 0.23, $c_f = 4.2 \cdot 10^{-3}$ (n = 2) in the coordinate $\eta = \rho u_\tau y/\mu = \varphi (c_f/2)^{1/2}$ Re, where $u_\tau = \sqrt{\tau_0/\rho} = U_\delta (c_f/2)^{1/2}$ (the points are the experimental data of [6]). The profile of the excess velocity $\Delta u = U_\delta - u$, calculated in the coordinate $\varphi^0 = y/\delta_1 = \varphi/\varphi_3$, is shown in Fig. 4b. In the external layer region this profile slightly deviates from the experimental points for $\varphi^0 \sim 0.4$, which is obviously a price for the approximation used for τ in the buffer zone. The exact expression for $\varphi_1 \le \varphi \le \varphi_2$, compatible with the expression for $\varphi \le \varphi_1$, is for $b' = cu/U_1 a_1^2$ of the form $\tau = \chi B(\varphi f''/f')^2$, where χ is the value of f'^2 at the point $\varphi = \varphi_1$; $B = c_f \rho U_\delta^2/2c^2m^2$. Then

$$\varphi\left(\frac{\varphi f''}{f'}\right)' + \chi^{-1}\beta f = 0. \tag{38}$$

For $f' \to 1$, $\chi \to 1$ Eqs. (23) and (38) are quite close to each other, with some of the deviation mostly related to the τ distribution.

NOTATION

ψ, wave function; x and y, longitudinal and transverse rectangular coordinates; t, time; U, absolute value of the forward speed in the layer; U_δ, external flow velocity; ρ, density (incompressible flow); a and b, wave amplitude and phase; ω, fluctuation frequency; δ, width of the fluctuation region; τ, shear stress; τ_0, friction stress at the wall; μ, viscosity coefficient; u and v, longitudinal and transverse components of the mean velocity; h, "quantum" parameter; and c_f, local friction coefficient.

LITERATURE CITED

1. Ya. A. Vagramenko, Application of Quantum-Mechanical Concepts to the Description of Jet Turbulence [in Russian], Cosmic Studies in the Ural, Naukova Dumka, Kiev (1981), No. 15, pp. 73-82.
2. Ya. A. Vagramenko, "Wave structure of the turbulence displacement zone," Inzh.-Fiz. Zh., 42, 734-740 (1982).
3. Ya. A. Vagramenko, "Wave and regular motion in an axisymmetric turbulent jet," in: Hydroaeromechanics and Elasticity Theory [in Russian], No. 29, Dnepropetrovsk (1982), pp. 3-15.
4. Ya. A. Vagramenko, Wave Structure of a Turbulent Boundary Layer [in Russian], Preprint No. 142, Institute of Problems in Mechanics, Academy of Sciences of the USSR, Moscow (1980).
5. A. A. Townsend, The Structure of Turbulent Shear Flow, 2nd ed., Cambridge University Press (1976).
6. H. Schlichting, Boundary Layer Theory, 7th ed., McGraw-Hill, New York (1979).